Absorption Areas and ”En-Route” Slot Allocation

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Frédéric Ferchaud
frederic.ferchaud@eurocontrol.int
PhD candidate at the University of Bordeaux 1
Supervised by MM. Castanet, Gavoille and Mosbah
LaBRI²
351, Cours de la Libération
33405 Talence CEDEX

Innovative Research Area-AE31 Office
91222 Brétigny sur Orge CEDEX

Abstract—We present an approach for aircraft slot allocation, the air traffic flow planning, that takes into account the stochastic dimension of traffic demand versus airspace capacity. Our approach intuitively suggests the use of null-slots to absorb the uncertainty resulting in delays of aircraft.

In this paper we study the trade-off between capacity lost and gain in reallocation time (reduction of delays).

I. INTRODUCTION

Air Traffic Flow Management (ATFM) is a service established with the objective of contributing to safe, orderly, and expeditious flow of air traffic [2].

In Europe, to realize a commercial flight between two airports, airline must submit a flight plan to the Central Flow Management Unit (CFMU), the organisation responsible for air traffic flow management to avoid traffic overload.

From flight plan information, traffic demand capacity balancing is to be made for the sectors where necessary (e.g. number of aircraft in a given period is higher than the capacity). In accordance to the principle of the “First Filed-First Served”, the system in charge of the regulation extracts all flights entering the specified airspace and sequences them into the order they would have arrived at the airspace in absence of any regulation.

Unfortunately, as a matter of fact, uncertain operational events (weather conditions, technical failure, waiting passengers,...) occur daily and disturb the CFMU planning, leading to safety sensitivity and sub-optimally used capacity; a disruption of initial plan.

To deal with this disruption, we introduce the absorption areas. An absorption area is a number of slots, left unfilled during the slot allocation process, allowing the absorption of such disturbances with least modification of the planning.

Finding the best configuration of the absorption areas corresponds to balancing two complementary objectives: minimizing “capacity loss” and minimizing the reallocation time (delays). Capacity loss correspond to the lost slots: the unused absorptions areas and the slots not taken by aircraft submitted to uncertainty.

In the section II of this paper we discuss the minimization of the load loss, then in section III the minimization of the reallocation time. Section IV concerns the relation between capacity loss and reallocation time in order to find the best slot allocation, then section V concerns a new strategy to improve the slot reallocation time.

II. ABSORPTION AREAS SIZE

A. Main Problem Definition

Let \( n \) be the number of available slots. Each slot can be fill with an aircraft, and the aircraft have a probability \( p \) to take its slot. If an aircraft does not take its slot, then it asks for a new one and its slot is lost.

In order to reallocate aircraft to other slots, we let some slots unfilled: the absorption areas.

One strategy to obtain the optimal size of absorption areas is to let all the unfilled slots at the end (cf Fig.1). Indeed, in this case, we will not have the case of an unfilled slot before a delayed aircraft. But this case increase the reallocation time. It is the reason why we do not take into account the reallocation time in the first part in order to find the best amount of absorption areas according to the probability of take-off.

Let \( z \) be the number of unfilled slots (Absorption Areas). We want to find the amount of aircraft to allocate in order to guarantee that all aircraft will take off in a given time period, representing \( n \) slots. One solution is to let the same amount of unfilled slot than allocated slot. But the rate occupation is low. We reach the smallest \( z \) which guarantee the previous constraint.
B. Slot Occupation Rate with and without Absorption Areas

Let \( C_{wo}(n, p) \) be the obtained throughput with \( n \) slots, with a probability \( p \) that an aircraft take off in its slot, and \( \mathbb{E}(C_{wo}(n, p)) \) the expected capacity without absorption areas.

Let \( C_w(n, z, p) \) be the obtained throughput with \( n \) slots, in which \( z \) are unfilled, and with a probability \( p \) that an aircraft take off in its slot, and \( \mathbb{E}(C_w(n, z, p)) \) the expected capacity with absorption areas.

\[
\mathbb{E}(C_{wo}(n, p)) = pn \\
\mathbb{E}(C_w(n, z, p)) = p(n - z) + \min(z, (1 - p)(n - z))
\]

Indeed, the unfilled slots will be used to reallocated delayed aircraft. We assume that an aircraft taking a reallocated slot take its new slot (an aircraft can not be delayed two times) \( C_w(n, z, p) \) is equal to the number of aircraft taking their allocated slot \( (p(n - z)) \) and those having take one in the absorption areas “\( z \)” i.e. either \( z \) if \( z \) too small, \( (1 - p)(n - z) \) else. From where we find \( z \):

\[
z = (1 - p)(n - z) = \frac{1 - p}{2 - p} n
\]

Thus we obtain:

\[
\mathbb{E}(C_w(n, p)) - \mathbb{E}(C_{wo}(n, p)) = n(1 - p)^2 \\
\mathbb{E}(C_w(n, p)) - \mathbb{E}(C_{wo}(n, p)) \geq 0
\]

![Graph](image.png)

Fig. 2. Obtained capacity with and without absorption areas, according to the take-off probability

Here we see the double interest of using absorption areas. First, if we respect the security constraints, we increase the occupation rate of the slots (see Fig 2), and moreover, we guarantee that all the allocated aircraft will take-off in a period of \( n \) slots. It is not the case for \( C_{wo} \): we allocated all slots to aircraft, and when an aircraft lost its slot, either we can not reallocated it during the regulation time (we need more slots!), either it take off as soon as it is ready, but the security constraints will not be respected.

Sadly, the previous results were obtained in the average case. It means, that we can have more delayed aircraft and we can not reallocate them. So we want to guarantee that in the most of the cases, the absorption areas will be big enough to avoid these cases.

For this we will use the Chernoff Bounds[1].

C. Chernoff-Capacity

Let \( X_i \) be the random variable define as:

\[
\Pr(X_i = 1) = p \\
\Pr(X_i = 0) = 1 - p
\]

We consider that each \( X_i \) is independent. Chernoff[1] says:

\[
\Pr\left(\sum_{i=1}^{n} X_i \leq (1 - \epsilon)\mathbb{E}\left(\sum_{i=1}^{n} X_i\right)\right) \leq e^{-\mathbb{E}(\sum_{i=1}^{n} X_i) \epsilon^2/2}
\]

Let \( X_i = 1 \) be the fact that an aircraft take its allocated slot, and \( X_i = 0 \) not. We assume:

\[
C_{wo}(n, p) = \sum_{i=1}^{n} X_i
\]

We have:

\[
\Pr\left( C_{wo}(n, p) \leq (1 - \epsilon)\mathbb{E}(C_{wo}(n, p)) \right) \leq e^{-\mathbb{E}(C_{wo}(n, p)) \epsilon^2/2}
\]

We pose \( \epsilon = \sqrt{2c \ln(C_{wo}(n, p))} \)

\[
\Pr\left( C_{wo}(n, p) \leq (1 - \sqrt{2c \ln(C_{wo}(n, p))})\mathbb{E}(C_{wo}(n, p)) \right) \leq \frac{1}{n^\alpha}
\]

\[
\Pr\left( C_{wo}(n, p) \geq pn - \sqrt{2c p n \ln(n)} \right) \geq 1 - \frac{1}{n^\alpha}
\]

Similarly, from Chernoff we obtain:

\[
\Pr\left( C_w(n, z, p) \geq p(n - z) - \sqrt{2cp(n - z) \ln(n - z)} \right) \geq 1 - \frac{1}{(n - z)^\alpha}
\]

Let \( A(n, z, p) \) be the number of aircraft missing their allocated slots on \( (n - z) \) allocated slots to aircraft.

\[
\Pr\left( A(n, z, p) \leq (1-p)(n-z) + \sqrt{2c(1-p)(n-z) \ln(n-z)} \right) \geq 1 - \frac{1}{(n - z)^\alpha}
\]

We find the optimal \( z \):

\[
z = (1 - p)(n - z) + \frac{2c(1-p)(n-z) \ln(n-z)}{2 - p}\sqrt{2c(1-p)(n-z) \ln(n-z)}
\]

\[
z = \frac{1 - p}{2 - p} n + \frac{1}{2 - p}\sqrt{2c(1-p)(n-z) \ln(n-z)}
\]

Let \( T_z(n) \) be the slots occupation rate for \( n \) given slots and \( z \) unfilled slots. We obtain:

\[
\Pr\left( T_z(n) = \frac{n - z}{n} \right) \geq 1 - \frac{1}{(n - z)^\alpha}
\]

For given \( n \) and \( \delta \).
\[ \delta = \frac{1}{n - z}, \] in the previous formula,
\[ c = -\frac{1}{\ln(n - z)}. \]

With \( \delta \) being the robustness of the slot allocation. If \( \delta = 0.1 \), it means that the slot allocation will fail in less than 10% of the cases, (we will have more delayed aircraft than absorption areas).

\section*{III. Reallocation Time}

We know the optimal \( z \) according to \( \delta \). Now, we want to know the average reallocation time (in number of slots) of the aircraft.

\subsection*{A. Delayed Aircraft Position}

Let \( \mathcal{A} = \{1, 2, ..., r\} \).

Let \( P_n(a) \) be the random variable defining the position of the \( i \)th delayed aircraft given that a aircraft will be delayed. \( P_n(a) \in \mathcal{A} \).

Let \( a \) given, so the distribution of \( P_n(a), \forall n \in \{1, 2, ..., a\} \) is given by:

\[ \begin{aligned}
&\text{Pr}(P_1(a) = k) = \binom{r - k}{a - 1} \binom{r}{a}^{-1} \\
&\text{Pr}(P_n(a) = k) = \sum_{j=1}^{k-1} \text{Pr}(P_{n-1}(a) = j) \binom{r - k - j}{a - n} \binom{r - j}{a - n + 1} \\
\end{aligned} \]

\textbf{Proof:}

The suit of delayed aircraft positions is given by the word \( P_1(a), P_2(a), ..., P_n(a) \) of length \( a \), like:

\[ \begin{align*}
&\{ P_n(a) \in \mathcal{A} \\
&P_n(a) > P_{n-1}(a) \\
\end{align*} \]

For \( n = 1 \), we have \( \binom{r}{a} \) possible words of length \( a \); the number of growing words beginning by \( k \) is the number of possible words of length \( a - 1 \) on the alphabet \( \{k + 1, k + 2, ..., r\} \), that is \( \binom{r - k}{a - 1} \), so:

\[ \text{Pr}(P_1(a) = k) = \binom{r - k}{a - 1} \binom{r}{a}^{-1} \]

We assume the rank \( n \) true, so:

\[ \begin{aligned}
&\text{Pr}(P_{n+1}(a) = k) = \sum_{j=1}^{k-1} \text{Pr}(P_{n+1}(a) = k, P_n(a) = j) \\
&\text{Pr}(P_{n+1}(a) = k) = \sum_{j=1}^{k-1} \text{Pr}(P_{n+1}(a) = k | P_n(a) = j) \times \text{Pr}(P_n(a) = j) \\
&\text{Pr}(P_{n+1}(a) = k | P_n(a) = j) \text{ corresponds to the probability of obtaining a word of length } a - n - 1 \text{ given that the first letter is } j < k, \text{ on the alphabet } \{j + 1, j + 2, ..., r\}, \text{ that is } \binom{r - j}{a - n} \text{ possibilities, and that the second letter be } a k. \text{ This is the number of possible words of length } a - n \text{ on the alphabet } \{k + 1, k + 2, ..., r\}, \text{ that is } \binom{r - k}{a - n - 1}. \]

\subsection*{B. Reallocation}

From the delayed aircraft position, we obtain the reallocation time. The used strategy is the following: we allocated \( r \) slots and we let \( z \) slots free. Each aircraft not taking its slot is reallocated in a “\( z \)” slot.

Let \( D_n(a) \) be the random variable standing for the number of slots that an aircraft must wait before being reallocated, knowing that we have \( a \) delayed aircraft.

\[ \begin{align*}
&\{ \text{Pr}(D_1(a) = k) = \text{Pr}(P_1(a) = r - k) \\
&\text{Pr}(D_n(a) = k) = \text{Pr}(P_n(a) = r - k + n - 1) \\
\end{align*} \]

Let \( X \) be the random variable standing for the number of delayed aircraft.

Let \( D_n \) standing for the average delay of the \( n \)th delayed aircraft not knowing the number of delayed aircraft:

\[ \text{Pr}(D_n = k) = \sum_{a=1}^{r} \text{Pr}(P_n(a) = r - k + n - 1) \text{Pr}(X = a) \]

From where we find the average reallocation time \( \mathbb{E}(D) \):

\[ \mathbb{E}(D) = \frac{1}{r} \sum_{n=1}^{r} \frac{1}{r - n + 1} \sum_{k=n}^{r} k \text{Pr}(D_n = k) \]

\textbf{IV. Slot Occupation According to the Reallocation Time}

We assume that the maximal number of delayed aircraft is \( z \), we have:

\[ \text{Pr}(D_n = k) = \sum_{a=1}^{z} \text{Pr}(P_n(a) = r - k + n - 1) \text{Pr}(X = a) \]

\[ \mathbb{E}(D) = \frac{1}{r} \sum_{n=1}^{r} \sum_{k=n}^{r} k \text{Pr}(D_n = k) \]

So, for \( \delta = 0.1 \) we obtain:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Reallocation time according to slot occupation rate for \( p = 0.75 \) (probability of take-off), for \( n \) varying from 5 to 100 and for a robustness of 90\%. The lower value of \( n \) are on the left, and the bigger on the right.}
\end{figure}

We see on the Fig.3 that the reallocation time increases exponentially with the slot occupation rate, and that due to the discretization of \( z \), we have scatter. Nevertheless we show...
that we have a bend. So we see that if we know the deal between the reallocation time and the slot occupation rate, we can know the best period length.

The Fig.4 show that we have the same type of results for $p$ varying from 0.5 to 0.95, with a greater growth of the reallocation time according to the slot occupation rate when $p$ decrease. Indeed, when $p$ is low, we have more delayed aircraft, so the average reallocation time is greater.

The obtained theoretical results were compared with those obtained in a simulated way (cf. Fig.5). The simulation is always a little bit more higher because we use a robustness on the number of delayed aircraft of 10%. It means that in 10% of the case we have more aircraft to reallocate than absorption areas, it is why the reallocation time is a little bit higher. So this figure show that our calculations are good.

![Fig. 4. Reallocation time according to slot occupation rate for $p$ (probability of take-off) varying from 0.5 to 0.95 and $n$ varying from 5 to 100. The first set of point on the left correspond to the results obtained with $p$ equal to 0.5, the second one for $p = 0.55$ and so forth until $p = 0.95$ and for a robustness of 90%](image)

![Fig. 5. Difference between simulation and theory for $p$ varying from 0.6 to 0.95.](image)

V. ABSORPTION AREAS DISTRIBUTION

We want to reduce the delayed aircraft reallocation time. For this we will divided the number of slots $n$ in periods composed of $r$ allocated slots followed by $z$ unfilled slots. Thus, the waiting time will be reduced, at the expense of the capacity.

Indeed, we want to have the same robustness. Previously, we were in the case with only one period. To keep the same robustness with $i$ periods, we need a robustness of $1 - \delta^{1/i}$ on each period in order to have a total robustness of $1 - \delta$ on the set of periods.

In a first time, we look on what we obtain for $n = 60$ and $n = 100$ with one, two and four periods. The obtained results are given by Fig.6.

On this figure we see that more $n$ is big, more gain we have on the reallocation time when we use two periods. This result show us that to use a big period is not a good solution, indeed, when we use two periods, we reduce the reallocation time without a big reduce on the capacity. But when $n$ is smaller (case $n = 60$) we see that we reduce the reallocation time, but the reduction of slot occupation rate is too big to use too small periods.

Moreover the choice to have an integer number of period does not permit us to find a real good solution: according to the case, we see that to have 2 or 4 periods will be the better choice, but we can easily suppose that the best period length is not necessary a multiple of $n$. It is why we will choose a period length not multiple of $n$.

![Fig. 6. Reallocation time according to the slot occupation rate for $n = 60$ and $n = 100$ with 1, 2 and 4 periods for a robustness of 90%. The higher point of each curve represent the result obtained with a single period, the middle one with two and the lower one with four, with the same robustness on $n$ slots. The curves on the left are obtained with $p = 0.5$, the next one with $p = 0.525$ and so forth until $p = 1$, the point of coordinate (1,0).](image)

![Fig. 7. Show us that the reallocation time decrease exponentially according to the period length, whatever is the probability that an aircraft take its slot. We explain this by the fact that the reallocation time depend of the period length, and the period length is given per $1+\frac{z}{r}$, so we expected a similarity with the function $\frac{1}{r}$.](image)

![Fig. 8. Corresponds to the result obtain with $n = 60$ according to the period length, for each set of point, on the left we have the obtained result with the smaller period. As expected, we obtain the same type of result as Fig.4. So we have to study](image)
the deal between the reallocation time and the slot occupation rate.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Reallocation time according to \( n \) for \( p \) varying from 0.6 to 0.95}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Reallocation time according to capacity, with a robustness of 90%}
\end{figure}

A. Best Period according to \( \alpha \)

Let \( \alpha \) be the random variable given the ratio between the cost of the reallocation time and the capacity:

The best period is obtained with the maximization of the following function:

\[
P = \max(T(r, z, n) - \alpha D(r, z, n))
\]

With \( D(r, z, n) \) being the occupation rate for \( n \) slots with periods of length \( r + z \), \( r \) being the number of allocated slots and \( z \) of unfilled.

And \( T(r, z, n) \) being the reallocation time in slots for \( n \) slots with periods of length \( r + z \), \( r \) being the number of allocated slots and \( z \) of unfilled.

According to \( \alpha \) we can find the best slot allocation which guarantee the safety conditions, and deal with the best configuration between the reallocation time and the occupation rate.

Sadly, due to discretization to find \( z \), smallest periods can not be considered because the gap between the Chernoff’s values and the integer \( z \) used is too big. For example, if we find a \( z \) equal to 1.5 for \( r \) equal to 3, then we have such allocation: \( r = 3 \), \( z = 2 \) on each period. And if we have 99 aircraft to allocate, then we put 16 more unfilled slots, than the result given by the Chernoff’s bound.

However we find range of good \( \alpha \) values, it means that the previous maximization does not give us the bigger period or the smallest one. Indeed We have three configuration of the previous function with fixed \( \alpha \). The first one is a logarithmic configuration which give us that we have to use the bigger period possible. The second one is the concave one which is the best case because we find a period length which is not the smaller one nor the bigger one. And the last one is a linear configuration in which we can not say anything except that all configuration can be used.

Moreover, due to discretization, we have many scatter on the alpha values. We have to solve this problem by using different length period in order to respect the Chernoff’s bound values.

\section{VI. Actual and Future Work}

A. \( \alpha \)

As previously said, we can find an optimization if \( \alpha \in [\alpha_1, \alpha_2] \). But due to discretization, we have lot of noise. To avoid this, we have to compare only the strategies in which \( r \) and \( z \) are close to be integer with the Chernoff’s Bound, in order to reduce the load loss. This work is in progress now.

Then we have to do the same study with the reused of slot and we will have the same problems due to discretization.

B. Reused of Lost Slots

1) Without taking into account the Reallocation Time: We did not consider that if an aircraft does not take its slot, its slot can be reused. In the reality, we can know this information enough time before the take-off (technical failure, delayed aircraft and having to take-off back...) to reused its allocated slot to another delayed aircraft (if it exists).

It means that for each aircraft we have three possibilities:

1) the aircraft take its allocated slot,
2) the aircraft can not take its slot and its slot can be reused,
3) the aircraft can not take its slot and its slot is lost.

Let \( p \) be the take-off probability, and \( q \) be the probability that we know enough time before the take-off time that an aircraft will not take its allocated slot (we can reused this slot).

The first case is an event with a probability \( p \), the second one the event with a probability \( (1-p)q \) and the third one the event with a probability \( (1-p)(1-q) \).

If we do not take into account slot reallocation time, we have found the best amount of absorption areas to apply to improve the management of the upper airspace.

This part is already done and presented in the paper for the 7th international conference on intelligent transportation system at Washington in 2004 [4].
2) With taking into account the Reallocation Time: As we have only three events possible for each slot, we can consider all the possibilities with $n$ slots as all the words of length $n$ on the alphabet $\{0, 1, 2\}$. It means that we have $3^n$ possibilities. For each word we can find the average reallocation time. For example, if $n = 10$ we can have the word 0010020102:

1) The first letter is equal to 0, it means that the first aircraft takes its slot.
2) The second letter is equal to 0, it means that the second aircraft takes its slot.
3) The third letter is equal to 1, it means that the third aircraft does not take its slot and its slot can be reused, but as we do not have any aircraft to reallocate, this slot is lost. The third aircraft is in the queue to be reallocated.
4) The fourth letter is equal to 0, it means that the fourth aircraft takes its slot.
5) The fifth letter is equal to 0, it means that the fifth aircraft takes its slot.
6) The sixth letter is equal to 2, it means that the sixth aircraft does not take its slot and its slot can not be reused. So we have in the queue the aircraft $\{A_3, A_6\}$.
7) The seventh letter is equal to 0, it means that the seventh aircraft take its slot.
8) The eighth letter is equal to 1, it means that the eighth aircraft does not take its slot and it can be reused by the first aircraft in the queue: $A_8$, now the queue is $\{A_6, A_8\}$.
9) The ninth letter is equal to 0, it means that the ninth aircraft takes its slot.
10) The tenth letter is equal to 2, it means that the tenth aircraft does not take its slot and its slot can not be reused, the queue is $\{A_6, A_8, A_{10}\}$.
11) The aircraft in the queues (it they exists) are reallocated in the absorption areas.

The probability that this event occurs is equal to:

$$p^0[(1-p)q]^2[(1-p)(1-q)]^2$$

From these informations we can find the reallocation time, sadly, to consider all strategies is complex, for example, for $n = 20$ we have to consider $3^{20}$ events, it is equal to 3486784401, which it takes approximately 25 GB of memory (only to develop all the events).

C. Space-Time Dependencies

We have solved the "En-Route" slot allocation problem only for a single sector. We have to take into account the space time dependencies to find a general algorithm which will improve the upper airspace management. Indeed, if an unfilled slot is taken by a delayed aircraft in a sector, and if this aircraft cross several regulated sectors, the unfilled slot must be in correlation otherwise, they were not be efficient, we can not used them.

REFERENCES